Approximate Controllability and Approximate Observability of Singular Distributed Parameter Systems

Zhaoqiang Ge, Xiaochi Ge, and Jifeng Zhang, *Fellow, IEEE*

***Abstract*—Necessary and sufficient conditions for the approximate controllability and approximate observability of a singular distributed parameter system are obtained in the sense of distributional solution. These general results are used to examine the approximate controllability and approximate observability of the Dzektser equation in the Theory of Seepage.**

***Index Terms*—Approximate controllability, approximate observability, distributional solution, singular distributed parameter systems.**

I. Introduction

Singular distributed parameter systems are also called degenerate evolution systems, generalized distributed parameter systems, and infinite-dimensional descriptor systems (e.g., [1]–[3]), etc. They appear in the study of heat flow, transmission lines, gas absorption, propagation of longitudinal waves in DNA molecules, and so on (e.g., [4], [5]). There is an essential distinction between singular and ordinary distributed parameter systems (e.g., [1]–[11]). Under disturbance, not only singular distributed parameter systems lose stability, but also great changes take place in their structure, such as leading to pulse behavior.

One of the most important problems for the study of singular distributed parameter systems is controllability. Many important results for the controllability of distributed parameter systems have been obtained (e.g., [10], Ch.4; [11], Ch.11), and the confluent Vandermonde matrices play an important role in it (e.g., [12]). But the results for the controllability of singular distributed parameter systems are very little. The approximate controllability, exact controllability, and exact null controllability for singular distributed parameter systems were studied in [1], [3], [13]–[16], respectively, in the sense of mild solution. The results show that the controllability of singular distributed parameter systems is quite different from distributed parameter systems. For example, in the case of the distributed parameter systems, approximate controllability is a dual of approximate observability. However, approximate controllability is not necessarily the dual of the approximate observability to singular distributed parameter systems in the sense of mild solution. For a singular distributed parameter system, pulse behavior may exist at initial time (e.g., [8]) which may reduce system performance and even damage the system. Therefore, we have to deal with the controllability of singular distributed parameter systems in the sense of distributional solution.

In this paper, the approximate controllability and approximate observability of a class of singular distributed parameter systems are studied in the sense of distributional solution. The dual principle is proved to be true for these two concepts.

Notations: Throughout the paper, X, Y, and U denote Hilbert spaces; $L(X,Y)$ denotes the space of all bounded linear operators from X into Y; $L(X) = L(X,X)$; $C_D(X,Y)$ denotes the set of all closed linear operators from X to Y whose domain is dense in X; $C_D(X) = C_D(X,X)$; $C^n(I,X)$ denotes the set of n times continuously differentiable X-valued functions on interval I; dom$A$ denotes the domain of operator A; $\overline{\text{ran}A}$ denotes the closure of ran$A$; A^* denotes the dual operator of A; $<\cdot,\cdot>_X$ denotes the inner product on the space X; $\|\cdot\|_X$ denotes the norm induced by the inner product on the space X; $L^2((0,T),U)$ denotes the class of Lebesgue measurable U-valued functions with $\int_0^T \|x(t)\|_U^2\, dt < +\infty$ and

$$X \times Y = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \in X, y \in Y \right\}.$$

The singular distributed parameter system

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $E \in L(X,Y)$, $A \in C_D(X,Y)$, and $B \in L(U,Y)$, is an abstract form of various partial differential equations and systems of equations which occur in heat flow, transmission lines, gas absorption, propagation of longitudinal waves in DNA molecules, motion of ground waters with a free surface, diffusive-convective system with limited manipulating variables, physically meaningful constraints, and so on (see, e.g., [4], [5], [17]–[19]).

For the sake of convenience, we introduce the following definition.

Definition 1: System (1) is called the regular system with order n (positive integer) if there exist Hilbert spaces X_1, X_2 and $P \in L(Y, X_1 \times X_2)$, $Q \in L(X_1 \times X_2, X)$, where P and Q are bijective, such that

$$PEQ = \begin{bmatrix} I_1 & 0 \\ 0 & N \end{bmatrix} \in L(X_1 \times X_2)$$

$$PAQ = \begin{bmatrix} K & 0 \\ 0 & I_2 \end{bmatrix} \in C_D(X_1 \times X_2)$$

and $PB = \left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right] \in L(U, X_1 \times X_2)$, where N is a nilpotent operator with order n ([8]); $K \in C_D(X_1)$ is the generator of the strongly continuous semigroup ([10, p. 15]); $I_k \in L(X_k)$ is the identical operator ($k = 1, 2$).

In this case, the operator P and Q transfer (1) into the following decoupled system on the Hilbert space $X_1 \times X_2$:

$$\dot{x}_1(t) = Kx_1(t) + B_1u(t) \tag{2}$$

$$N\dot{x}_2(t) = x_2(t) + B_2u(t) \tag{3}$$

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Z. Q. Ge is with the School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China (e-mail: gezqjd@mail.xjtu.edu.cn).

X. C. Ge is with the Cass Business School, City, University of London, London EC1Y 8TE, U.K. (e-mail: Xiaochi.Ge@cass.city.ac.uk).

J. F. Zhang is with the Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Science, Beijing 100190, China (e-mail: jif@iss.ac.cn).

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where $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] = Q^{-1}x$, $x_1 \in X_1$, $x_2 \in X_2$. The system represented by (2)–(3) is called the standard form of regular system (1).

From [7, pp. 135–138], we obtain that if A is a strongly (E, p)-radial operator in (1), then (1) is the regular singular distributed parameter system with finite order n, and $n \le p + 1$.

Here we recall the definitions of (E, p)-radial and strongly (E, p)-radial operators, respectively.

An operator A is called (E, p)-radial if

i) $\forall \mu \in \mathbb{R}_+ \ \mu \in \{\lambda \in \mathbb{C} : (\lambda E - A)^{-1} \in L(Y, X)\}$

ii) $\exists K \in \mathbb{R}_+ \ \forall \mu_k \in \mathbb{R}_+ (k = 0, 1, \ldots, p) \ \forall n \in \mathbb{N}$ $\max\{\|[\prod_{k=0}^{p}((\mu_k E - A)^{-1}E)]^n\|_{L(X)}$

$$\left\| \left[\prod_{k=0}^{p} E(\mu_k E - A)^{-1} \right]^n \right\|_{L(Y)} \Bigg\} \le K / \prod_{k=0}^{p} \mu_k^n.$$

An operator A is called strongly (E, p)-radial if it is (E, p)-radial, and there exists a subspace Y_0 dense in Y such that

$$\left\| A(\lambda E - A)^{-1} \left[\prod_{k=0}^{p} E(\mu_k E - A)^{-1} \right] f \right\|_Y \le \text{const}(f) \left[\lambda \prod_{k=0}^{p} \mu_k \right]^{-1}$$

$\forall f \in Y_0$ for every $\lambda, \mu_0, \mu_1, \ldots, \mu_p \in \mathbb{R}_+$, and

$$\left\| \left[\prod_{k=0}^{p} ((\mu_k E - A)^{-1} E) \right] (\lambda E - A)^{-1} \right\|_{L(Y,X)} \le K \left[\lambda \prod_{k=0}^{p} \mu_k \right]^{-1}$$

$\forall \lambda, \mu_0, \mu_1, \ldots, \mu_p \in \mathbb{R}_+$. For more details, see [7, Ch. 2].

In addition, from [7, pp. 119–138], we see that many systems are regular, such as the Navier–Stokes equation; the robotic system; degenerate system of ordinary differential equations, which is the generalization of the well-known Leontief's system of inter-industry balance; the equation modeling the free surface evolution of a filtered fluid; among others. Subsystem (2) is a classical system in control theory. The properties of (3) determine the peculiarities of (1). For example, it is known that controls from the class $C^{n-1}([0, +\infty), U)$ must be used to solve (3) in the weak sense (e.g., [16]). This paper investigates the approximate controllability of (1) under some additional hypotheses, or, equivalently, of (2)–(3) and the corresponding approximate observability. In Section II, the definition of approximate controllability of (2)–(3) is introduced. Some necessary and sufficient conditions concerning the approximate controllability are given. In Section III, the concept of approximate observability is introduced. Some necessary and sufficient conditions concerning this concept are obtained. The dual principle of the approximate controllability and approximate observability is given in Section IV. In Section V, the general results obtained are used to examine the approximate controllability and approximate observability of the Dzektser equation in the Theory of Seepage. Finally, in the last section, we summarize our results.

Here we give several auxiliary results. As in the distributed parameter systems case, approximate controllability is described by the trajectory of the system. Since the trajectories of two equivalent systems are related by a constant revertible bounded linear operator, the approximate controllability is invariant under system equivalence. Note that every regular singular distributed parameter system is equivalent to (2)–(3), without loss of generality; in the following, we assume that system (1) is of the form (2)–(3).

Theorem 1: ([8]) Suppose that (2)–(3) is the standard form of a regular system with order n, $u \in C^{n-1}([0, +\infty), U)$, and there exist constants $M > 0$, $a > 0$ such that

$$\|u^{(k)}(t)\|_U \le Me^{at}, k = 0, 1, \ldots, n - 1.$$

Then, for each initial $\left[\begin{smallmatrix} x_{10} \\ x_{20} \end{smallmatrix}\right] \in X_1 \times X_2$, there exists a unique distributional solution of (2)–(3)

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{Kt}x_{10} + \int_0^t e^{K(t-\tau)}B_1 u(\tau)d\tau \\ x_{2\text{pulse}}(t) + x_{2\text{normal}}(t) \end{bmatrix} \tag{4}$$

where e^{Kt} denotes the strongly continuous semigroup generated by K,

$$x_{2\text{pulse}}(t) = -\sum_{k=1}^{n-1} N^k \delta^{(k-1)}(t) \left[x_{20} + \sum_{k=0}^{n-1} N^k B_2 u^{(k)}(0) \right]$$

$x_{2\text{normal}}(t) = -\sum_{k=0}^{n-1} N^k B_2 u^{(k)}(t)$, $\delta(t)$ is the Dirac function, $\delta^{(k)}(t)$ is the kth derivative of $\delta(t)$.

Here we recall the definition of $\delta^{(k)}(t)$.

Let $D(\mathbb{R})$ denote the space of infinite times continuously differentiable maps from $\mathbb{R}$ to $\mathbb{R}$ with compact support. $\delta^{(k)}(t)$ is defined by $\int_{-\infty}^{+\infty} \delta^{(k)}(\tau)g(\tau)d\tau = (-1)^k g^{(k)}(0)$ for every $g \in D(\mathbb{R})$. For more details, see [20, Ch.2].

It is well known that a mild solution $x_1(t)$ of (2) is expressible for $x_{10} \in X_1$, $u \in L^2((0, T), U)$ by the formula

$$x_1(t) = e^{Kt}x_{10} + \int_0^t e^{K(t-\tau)}B_1 u(\tau)d\tau \tag{5}$$

where the integral is understood in the sense of Bochner ([10, p. 104]).

Note that the first line of the matrix in (4), which gives a solution of (2)–(3), is a mild solution of (2), while the second line, which is a sum over k, is a distributional solution of (3), and

$$x_2(t) = -\sum_{k=0}^{n-1} N^k B_2 u^{(k)}(t), t > 0. \tag{6}$$

In the following discussions, we shall assume by default that the solutions are distributional.

Definition 2: A number $\lambda \in \mathbb{C}$ is called the E-eigenvalue of the operator A if there exists a vector $x \neq 0$ such that $\lambda Ex = Ax$. Such a vector x is called the E-eigenvector of the operator A corresponding to the E-eigenvalue λ.

It is easily verified that the E-eigenvectors corresponding to the same E-eigenvalue form a subspace of X.

II. Approximate Controllability

Consider the system described by (2)–(3). The extension of the concept of approximate controllability from distributed parameter systems to singular distributed parameter systems is as follows.

Definition 3: System (2)–(3) is called approximately controllable on $[0, T]$ (for some finite $T > 0$) if, for any state $x_T \in X_1 \times X_2$, any initial state $x_0 \in X_1 \times X_2$ and any $\epsilon > 0$, there exists a control $u \in C^{n-1}([0, T], U)$ such that the solution $x(t)$ of the system satisfies $\|x(T) - x_T\|_{X_1 \times X_2} < \epsilon$.

Our purpose here is to establish necessary and sufficient conditions for the approximate controllability of (2)–(3) with bounded operators B_1 and B_2.

As for the approximate controllability of (2), we have the following results.

Theorem 2: ([10, p. 148]) Subsystem (2) is approximately controllable on $[0, T]$ if and only if any one of the following conditions hold:

i) $\int_0^T e^{K\tau} B_1 B_1^* e^{K^*\tau} d\tau > 0$

ii) $B_1^* e^{K^*\tau} z = 0$ on $[0, T] \Rightarrow z = 0$.

According to Theorem 2, we obtain the following proposition.

Proposition 1: Subsystem (2) is approximately controllable on $[0, T]$ if and only if

$$G(f, T) = \int_0^T f^2(\tau) e^{K\tau} B_1 B_1^* e^{K^*\tau} d\tau > 0$$

for any polynomial $f(\tau) \in \mathbb{R}$ not identically zero and

$$\overline{\text{ran} G(f, T)} = X_1.$$

Proof: We only need to prove $\ker G(f,T) = \ker G(1,T)$ by the proof of Theorem 2 ([10, p. 149]). Since $f(\tau)$ can have only finitely many zeros in the interval $[0, T]$, it follows that $f(\tau) B_1^* e^{K^*\tau} z = 0$ on $[0, T]$ if and only if $B_1^* e^{K^*\tau} z = 0$ on $[0, T]$ if and only if $\ker G(f,T) = \ker G(1,T)$. ■

As for the approximate controllability of (3), we have the following theorem.

Theorem 3: Subsystem (3) is approximately controllable on $[0, T]$ if and only if

$$\overline{\text{ran}[B_2 \quad NB_2 \quad \cdots \quad N^{n-1}B_2]} = X_2. \tag{7}$$

Proof: Necessity. Approximate controllability of (3) on $[0, T]$ implies that for any state $x_{2T} \in X_2$, any initial state $x_{20} \in X_2$, and any $\epsilon > 0$, there exists a control $u \in C^{n-1}([0, T], U)$ such that the solution given by (6) satisfies $\|x_2(T) - x_{2T}\|_{X_2} < \epsilon$. Therefore, (7) is true.

Sufficiency: Since (7) holds, for any state $x_{2T} \in X_2$, any initial state $x_{20} \in X_2$, and any $\epsilon > 0$, there exist $\beta_k \in U, k = 0, 1, \ldots, n-1$ such that $\| - \sum_{k=0}^{n-1} N^k B_2 \beta_k - x_{2T}\|_{X_2} < \epsilon$. By (6), it follows that, for any $t > 0$, the corresponding solution is determined only by the value $u^{(k)}(t), k = 0, 1, \ldots, n-1$. Therefore, if a control $u(t)$ satisfies

$$u^{(k)}(T) = \beta_k, k = 0, 1, \ldots, n-1 \tag{8}$$

then, (6) yields that $x_2(T) = -\sum_{k=0}^{n-1} N^k B_2 \beta_k$. In order to build a control $u \in C^{n-1}([0, T], U)$ satisfying (8), let

$$u(t) = \sum_{k=0}^{n-1} \frac{(t-T)^k}{k!} \beta_k$$

then, (8) holds true. Hence, (3) is approximately controllable. ■

Now we discuss the approximate controllability of (2)–(3).

Theorem 4: System (2)–(3) is approximately controllable on $[0, T]$ if and only if both (2) and (3) are approximately controllable on $[0, T]$.

Proof: The necessity is obvious. We only need to prove the sufficiency. Assume $x_{10}, x_{1T} \in X_1, x_{20}, x_{2T} \in X_2$, and $\epsilon > 0$. We have to find $u \in C^{n-1}([0, T], U)$ such that

$$x_1(t) = e^{Kt} x_{10} + \int_0^t e^{K(t-\tau)} B_1 u(\tau) d\tau$$

$$x_2(t) = -\sum_{k=0}^{n-1} N^k B_2 u^{(k)}(t), t > 0$$

and

$$\|x_1(T) - x_{1T}\|_{X_1} < \epsilon, \|x_2(T) - x_{2T}\|_{X_2} < \epsilon. \tag{9}$$

We choose $u(t) = u_1(t) + u_2(t)$. Thus,

$$x_1(t) = e^{Kt} x_{10} + \int_0^t e^{K(t-\tau)} B_1 u_1(\tau) d\tau + \int_0^t e^{K(t-\tau)} B_1 u_2(\tau) d\tau$$

$$x_2(t) = -\sum_{k=0}^{n-1} N^k B_2 u_1^{(k)}(t) - \sum_{k=0}^{n-1} N^k B_2 u_2^{(k)}(t).$$

We choose $u_1(t)$ to be of the form

$$u_1(t) = t^n (t-T)^n v(t) \tag{10}$$

for some $v \in C^n([0, T], U)$. Thus, $u_1^{(k)}(T) = 0$, if $k < n$. By Theorem 3, there exists $u_2 \in C^{n-1}([0, T], U)$ such that

$$\left\| -\sum_{k=0}^{n-1} N^k B_2 u_2^{(k)}(T) - x_2(T) \right\|_{X_2} < \epsilon.$$

From Proposition 1, for any polynomial $f \in \mathbb{R}$ not identically zero, there exists $y \in X_1$, such that

$$\left\| \int_0^T f^2(\tau) e^{K(T-\tau)} B_1 B_1^* e^{K^*(T-\tau)} y d\tau - \left[x_{1T} - e^{KT} x_{10} - \int_0^T e^{K(T-\tau)} B_1 u_2(\tau) d\tau \right] \right\|_{X_1} < \epsilon. \tag{11}$$

Let $f(\tau) = \tau^n (\tau - T)^n$, $v(\tau) = f(\tau) B_1^* e^{K^*(T-\tau)} y$. Then, by (10) and (11), $u_1(\tau) = f^2(\tau) B_1^* e^{K^*(T-\tau)} y$ and

$$\left\| \int_0^T f^2(\tau) e^{K(T-\tau)} B_1 B_1^* e^{K^*(T-\tau)} y d\tau - \left[x_{1T} - e^{KT} x_{10} - \int_0^T e^{K(T-\tau)} B_1 u_2(\tau) d\tau \right] \right\|_{X_1} < \epsilon.$$

Thus, (9) is true. Therefore, (2)–(3) is approximately controllable on $[0, T]$. ■

III. Approximate Observability

In this section, we introduce the dual concept-approximate observability. This type of observability is concerned with the ability to reconstruct the state from the system output. Therefore, different from Section II, in this section, the system to be considered is the following form:

$$\begin{cases} \begin{bmatrix} I_1 & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} K & 0 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ y(t) = [D_1 \quad D_2] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases} \tag{12}$$

where D_1 and D_2 are bounded linear operators from X_1 and X_2 to Hilbert space Z, respectively. The two subsystems of (12) are assumed to be

$$\begin{cases} \dot{x}_1(t) = K x_1(t) \\ y_1(t) = D_1 x_1(t) \end{cases} \tag{13}$$

and

$$\begin{cases} N \dot{x}_2(t) = x_2(t) \\ y_2(t) = D_2 x_2(t). \end{cases} \tag{14}$$

Definition 4: System (12) is called approximately observable on $[0,T]$ (for some finite $T>0$) if $\ker(D_1 e^{Kt}) \equiv \{0\}\ \forall t \in [0,T]$ and if $y_2(t) \equiv 0, t \in [0,T]$ then $x_2(0) = 0$.

Clearly, Definition 4 reduces to the approximate observability in distributed parameter system theory when system (12) is a distributed parameter system. By Definition 4, we can obtain the following theorem.

Theorem 5: Let (13) and (14) be two subsystems of the regular system (12).

i) ([10, p.156]) Subsystem (13) is approximately observable on $[0,T]$ if and only if $D_1 e^{Kt} z = 0$ on $[0,T] \Rightarrow z = 0$.

ii) Subsystem (14) is approximately observable on $[0,T]$ if and only if

$$\overline{\mathrm{ran}[D_2^* \quad N^* D_2^* \quad \cdots \quad (N^*)^{n-1} D_2^*]} = X_2.$$

iii) System (12) is approximately observable on $[0,T]$ if and only if both (13) and (14) are approximately observable on $[0,T]$.

Proof of conclusion (ii): By Definition 4, subsystem (14) is approximately observable on $[0,T]$ means that $x_2(0) = 0$ if and only if $y_2(t) \equiv 0$, $t \in [0,T]$. By (4), we have

$$\begin{cases} y_2(t) = -\sum_{k=1}^{n-1} \delta^{(k-1)}(t) D_2 N^k x_2(0), t \in (0,T] \\ y_2(0) = D_2 x_2(0). \end{cases}$$

According to the linear independencies of $\delta^{(k-1)}(t)$ $(k = 1, 2, \dots, n-1)$, we have that

$$\begin{cases} y_2(t) = -\sum_{k=1}^{n-1} \delta^{(k-1)}(t) D_2 N^k x_2(0) \equiv 0, t \in (0,T] \\ y_2(0) = D_2 x_2(0) = 0 \end{cases}$$

if and only if $D_2 N^k x_2(0) = 0$ $(k = 0, 1, 2, \dots, n-1)$. Therefore, $x_2(0) = 0$ if and only if

$$\ker \begin{bmatrix} D_2 \\ D_2 N \\ \vdots \\ D_2 N^{n-1} \end{bmatrix} = \{0\}.$$

By Theorem A.3.16 of [10], this is equivalent to

$$\overline{\mathrm{ran}[D_2^* \quad N^* D_2^* \quad \cdots \quad (N^*)^{n-1} D_2^*]} = X_2.$$

Proof of conclusion (iii): By Definition 4, system (12) is approximately observable on $[0,T]$ means that $\left[\begin{smallmatrix} x_1(0) \\ x_2(0) \end{smallmatrix}\right] = 0$ if and only if $y(t) \equiv 0$, $t \in [0,T]$. According to (4), we have

$$\begin{cases} y_1(t) = D_1 e^{Kt} x_1(0), y_2(0) = D_2 x_2(0) \\ y_2(t) = -\sum_{k=1}^{n-1} \delta^{(k-1)}(t) D_2 N^k x_2(0), t \in (0,T] \\ y(t) = y_1(t) + y_2(t). \end{cases}$$

In view of the special forms of $y_1(t)$ and $y_2(t)$, we know that $y(t) \equiv 0$, $t \in [0,T]$ if and only if $y_1(t) \equiv 0$ and $y_2(t) \equiv 0$, $t \in [0,T]$. Therefore, by the proofs of conclusions (i) and (ii) of the theorem, $\left[\begin{smallmatrix} x_1(0) \\ x_2(0) \end{smallmatrix}\right] = 0$ if and only if $D_1 e^{Kt} x_1(0) = 0$ on $[0,T] \Rightarrow x_1(0) = 0$, and

$$\overline{\mathrm{ran}[D_2^* \quad N^* D_2^* \quad \cdots \quad (N^*)^{n-1} D_2^*]} = X_2.$$

Thus, the third conclusion holds in view of the first two conclusions of the theorem. ∎

IV. The Dual Principle

In this section, we deal with the dual principle for singular distributed parameter system. Let us first introduce the dual system of a regular singular distributed parameter system in the form of

$$\begin{cases} \begin{bmatrix} I_1 & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} \\ \quad = \begin{bmatrix} K & 0 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \\ y(t) = [D_1 \quad D_2] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \end{cases} \tag{15}$$

The two subsystems of (15) are

$$\begin{cases} \dot{x}_1(t) = K x_1(t) + B_1 u(t) \\ y_1(t) = D_1 x_1(t) \end{cases} \tag{16}$$

and

$$\begin{cases} N \dot{x}_2(t) = x_2(t) + B_2 u(t) \\ y_2(t) = D_2 x_2(t). \end{cases} \tag{17}$$

Definition 5: The following system

$$\begin{cases} \begin{bmatrix} I_1 & 0 \\ 0 & N^* \end{bmatrix} \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} \\ \quad = \begin{bmatrix} K^* & 0 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} D_1^* \\ D_2^* \end{bmatrix} v(t) \\ w(t) = [B_1^* \quad B_2^*] \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \end{cases} \tag{18}$$

is called the dual system of (15).

If (18) is the dual system of (15), then the two subsystems of (18)

$$\begin{cases} \dot{z}_1(t) = K^* z_1(t) + D_1^* v(t) \\ w_1(t) = B_1^* z_1(t) \end{cases} \tag{19}$$

and

$$\begin{cases} N^* \dot{z}_2(t) = z_2(t) + D_2^* v(t) \\ w_2(t) = B_2^* z_2(t). \end{cases} \tag{20}$$

are the dual systems of (16) and (17), respectively.

The following dual principle reveals the relation between the approximate controllability (approximate observability) of system (15) and the approximate observability (approximate controllability) of its dual system (18).

Theorem 6: System (15) is approximately controllable (approximately observable) on $[0,T]$ if and only if its dual system (18) is approximately observable (approximately controllable) on $[0,T]$.

Proof: It follows from Theorems 2–5 that the following equivalence relations hold: System (15) is approximately controllable if and only if

$$\begin{cases} B_1^* e^{K^* \tau} \alpha = 0 (\tau \in [0,T]) \Rightarrow \alpha = 0 \\ \overline{\mathrm{ran}[B_2 \quad N B_2 \quad \cdots \quad N^{n-1} B_2]} = X_2 \end{cases}$$

if and only if subsystems (19) and (20) are approximately observable on $[0,T]$ and if and only if system (18) is approximately observable on $[0,T]$. ∎

Remark 1: If $B_1 \in L(U, D(K^*)')$ is an admissible control operator for e^{Kt} ([11, pp. 355–356]), $B_2 \in L(U, X_2)$, $D_1 \in L(D(K), Z)$ is an admissible observation operator for e^{Kt} ([11, p. 173]) and $D_2 \in L(X_2, Z)$, the results in Sections II–IV are still valid, where $D(K)$ denotes $\mathrm{dom} K$ with the norm $\| \cdot \|_1$ (for more details, see [11, p. 173]).

V. Illustrative Example

In this section, we discuss the approximate controllability and approximate observability of the Dzektser equation.

Consider the Dzektser equation, which describes the evolution of the free surface of Seepage liquid (see, e.g., [17])

$$\left(1+\frac{\partial^2}{\partial \xi^2}\right)\frac{\partial}{\partial t}x(\xi,t)=\left(\frac{\partial^2}{\partial \xi^2}+2\frac{\partial^4}{\partial \xi^4}\right)x(\xi,t)+u(t)$$
$$(\xi,t)\in(0,\pi)\times[0,+\infty) \tag{21}$$

$$x(0,t)=\frac{\partial^2}{\partial \xi^2}x(0,t)=x(\pi,t)=\frac{\partial^2}{\partial \xi^2}x(\pi,t)=0$$
$$t\in[0,+\infty), x(\xi,0)=x_0(\xi), \xi\in(0,\pi) \tag{22}$$
$$y(\xi,t)=x(\xi,t),(\xi,t)\in(0,\pi)\times[0,+\infty). \tag{23}$$

Let

$$X=\{x\in W^{2,2}(0,\pi):x(0)=0,x(\pi)=0\}$$
$$Y=L^2((0,\pi),\mathbb{R})$$
$$E=1+\frac{\partial^2}{\partial\xi^2}, A=\frac{\partial^2}{\partial\xi^2}+2\frac{\partial^4}{\partial\xi^4}$$
$$\mathrm{dom}A=\{x\in W^{4,2}(0,\pi):x(0)=x''(0)=x(\pi)=x''(\pi)=0\}$$
$$(x(t))(\xi)=x(\xi,t),(Bu)(\xi)=bu,\xi\in(0,\pi),u\in U=\mathbb{R}$$

$b=1\in Y$, where the meanings of Sobolev spaces $W^{2,2}(0,\pi)$ and $W^{4,2}(0,\pi)$ are the same as in [21, Ch. 3]. Then, $E\in L(X,Y)$, $A\in C_D(X,Y)$ and Dzektser (21)–(23) can be reduced to the following system:

$$E\dot{x}(t)=Ax(t)+Bu(t),x(0)=x_0 \tag{24}$$
$$y(t)=x(t). \tag{25}$$

It is easily checked that $\sin(k\xi)$ is the E-eigenvector of the operator A corresponding to E-eigenvalue $-k^2(1+\frac{k^2}{k^2-1})$ of the operator A $(k=2,3,\ldots)$;

$$E\sin\xi=0; A\sin\xi=\sin\xi$$
$$E\sin(k\xi)=(1-k^2)\sin(k\xi)(k=2,3,\ldots)$$
$$A\sin(k\xi)=(2k^4-k^2)\sin(k\xi)(k=2,3,\ldots)$$

and

$$1=\sum_{k=1}^{+\infty}\frac{<1,\sin(k\xi)>_Y}{<\sin(k\xi),\sin(k\xi)>_Y}\sin(k\xi)$$
$$=\frac{4}{\pi}\sin\xi+\sum_{k=2}^{+\infty}\frac{4\sin((2k-1)\xi)}{(2k-1)\pi},\xi\in(0,\pi).$$

Let X_1 be the closure of the subspace

$$\mathrm{span}\{\sin(k\xi):k=2,3,\ldots\}$$
$$=\left\{x_1\in X:\exists a_k\in\mathbb{R},k=2,3,\ldots,x_1=\sum_{k=2}^{+\infty}a_k\sin(k\xi)\right.$$
$$\left.\times\|x_1\|_X<+\infty\right\}$$

in the norm of the space X; $X_2=\mathrm{span}\{\sin\xi\}$. Then, X_2 is one-dimensional. Let E_1 and A_2 denote the restrictions of E and A on X_1 and X_2, respectively. Then,

$$K\sin(k\xi)=E_1^{-1}A\sin(k\xi)$$
$$=-k^2\left(1+\frac{k^2}{k^2-1}\right)\sin(k\xi),k=2,3,\ldots$$
$$\mathrm{dom}K=\mathrm{span}\{\sin(k\xi):k=2,3,\ldots\}$$
$$b_1=E_1^{-1}\sum_{k=2}^{+\infty}\frac{<1,\sin(k\xi)>_Y}{<\sin(k\xi),\sin(k\xi)>_Y}\sin(k\xi)$$
$$=E_1^{-1}\left(1-\frac{4}{\pi}\sin\xi\right)$$
$$=\sum_{k=2}^{+\infty}\frac{4}{(2k-1)\pi}E_1^{-1}\sin((2k-1)\xi)$$
$$=\sum_{k=2}^{+\infty}\frac{4\sin((2k-1)\xi)}{[1-(2k-1)^2](2k-1)\pi}$$
$$b_2=A_2^{-1}\frac{<1,\sin\xi>_Y}{<\sin\xi,\sin\xi>_Y}\sin\xi=\frac{4}{\pi}\sin\xi$$

the regular standard form of (24)–(25) is

$$\dot{x}_1(t)=Kx_1(t)+B_1u(t),x_1(0)=x_{10} \tag{26}$$
$$0=x_2(t)+B_2u(t),x_2(0)=x_{20} \tag{27}$$
$$y(t)=x(t)=\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix} \tag{28}$$

where $[\begin{smallmatrix}x_1(t)\\x_2(t)\end{smallmatrix}]\in X_1\times X_2$, $K\in C_D(X_1)$, $N=0$ in (27), $B_1u=b_1u$, $B_2u=b_2u$. Since $X_2=\overline{\mathrm{ran}[B_2]}$, conditions of Theorems 3 and 4 guarantee that (21)–(22) is approximately controllable on $[0,T]$ for some $T>0$ if and only if (26) is approximately controllable. It is obvious that the strongly continuous semigroup associated with (26) is given by

$$e^{Kt}\eta=\sum_{k=2}^{+\infty}e^{-k^2\left(1+\frac{k^2}{k^2-1}\right)t}\frac{<\eta,\sin(k\xi)>_X}{<\sin(k\xi),\sin(k\xi)>_X}\sin(k\xi).$$

Since $e^{Kt}=e^{K^*t}$, by (ii) of Theorem 2, the condition for approximate controllability is that there exists $T>0$ such that $B_1^*e^{K^*t}\eta=0$ on $[0,T]\Rightarrow\eta=0$. In fact, for any $T>0$, if $B_1^*e^{K^*t}\eta=0$ on $[0,T]$, then

$$e^{K^*t}\eta=\sum_{k=2}^{+\infty}e^{-k^2\left(1+\frac{k^2}{k^2-1}\right)t}\frac{<\eta,\sin(k\xi)>_X}{<\sin(k\xi),\sin(k\xi)>_X}\sin(k\xi)$$
$$=0,\xi\in(0,\pi),\xi\neq\arcsin\frac{\pi}{4}.$$

Since sine series

$$\sum_{k=2}^{+\infty}e^{-k^2\left(1+\frac{k^2}{k^2-1}\right)t}\frac{<\eta,\sin(k\xi)>_X}{<\sin(k\xi),\sin(k\xi)>_X}\sin(k\xi)$$

is uniformly convergent on $[0,\pi]$ for every $t\in(0,T]$, we have

$$\sum_{k=2}^{+\infty}e^{-k^2\left(1+\frac{k^2}{k^2-1}\right)t}\left(\frac{<\eta,\sin(k\xi)>_X}{<\sin(k\xi),\sin(k\xi)>_X}\right.$$
$$\left.\times<\sin(k\xi),\sin(m\xi)>_X\right)=0.$$

By the orthogonality of the sine function system $\{\sin(k\xi) : k = 2, 3, \ldots\}$ on $[0, \pi]$, we get $< \eta, \sin(m\xi) >_X = 0, m = 2, 3, \ldots$. Since $\eta \in X_1$, we have $< \eta, \sin\xi >_X = 0$, i.e., $\eta = 0$. Hence, the Dzektser equation (21)–(22) is approximately controllable on $[0, T]$ for any $T > 0$.

From Definition 4 and (28), (26)–(28) is approximately observable on $[0, T]$ for any $T > 0$. Hence, Dzektser equation (21)–(23) is approximately observable on $[0, T]$ for any $T > 0$.

VI. Conclusion

We have defined approximate controllability and approximate observability and proved corresponding necessary and sufficient conditions for regular singular distributed parameter systems. The obtained results are very important and convenient for studying the approximate controllability and approximate observability of singular distributed parameter systems. An illustrative example was given, which shows the effectiveness of Theorems 4 and 5. For a specific singular distributed parameter system, appropriate controllability can be defined according to the needs of various optimal control problems.

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References

[1] A. Favini, "Controllability conditions of linear degenerate evolution systems," *Appl. Math. Optim.*, vol. 6, pp. 153–167, 1980.

[2] Z. Q. Ge, G. T. Zhu, and Y. H. Ma, "Spectrum distribution of the second order generalized distributed parameter systems," *Appl. Math. Mech.*, vol. 24, pp. 1224–1232, 2003.

[3] T. Reis, "Controllability and observability of infinite dimensional descriptor systems," *IEEE Trans. Autom. Control*, vol. 53, no. 4, pp. 929–940, May 2008.

[4] W. Marszalek, "On solving of some heat exchangers problems via image processing equations," *Archiwum Termodynamiki*, vol. 8, pp. 55–72, 1987.

[5] G. Chen and G. Zhang, "Initial boundary value problem for a system of generalized IMBq equations," *Math. Methods Appl. Sci.*, vol. 27, pp. 497–518, 2004.

[6] A. Favini and A. Yagi, *Degenerate Differential Equations in Banach Spaces*. New York, NY, USA: Marcel Dekker, 1999.

[7] G. A. Sviridyuk and V. E. Fedorov, *Linear Sobolev Type Equations and Degenerate Semigroups of Operators* (Inverse and Ill-Posed Problems Series). Utrecht, The Netherlands: VSP, 2003.

[8] Z. Q. Ge, F. Liu, and D. X. Feng, "Pulse controllability of singular distributed parameter systems," *SCI China Inf. Sci.*, vol. 62, pp. 049201:1–049201:3, 2019.

[9] D. Doleckv and D. L. Russell, "A general theory of observation and control," *SIAM J. Control Optim.*, vol. 15, pp. 185–220, 1977.

[10] R. F. Curtain and H. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*. New York, NY, USA: Springer-Verlag, 1995.

[11] M. Tucsnak and G. Weiss, *Observation and Control for Operator Semigroups*. Berlin: Birkhuser, 2009.

[12] J. S. Respondek, "Incremental numerical recipes for the high efficient inversion of the confluent Vandermonde matrices," *Comput. Math. Appl.*, vol. 71, pp. 489–502, 2016.

[13] V. Barbu and A. Favini, "Control of degenerate differential systems," *Control Cybern.*, vol. 28, pp. 397–420, 1999.

[14] M. R. A. Moubarak, "Exact controllability and spectrum assignment for infinite dimensional singular systems," *Publ. Math. Debrecen*, vol. 54, pp. 11–19, 1999.

[15] Z. Q. Ge, G. T. Zhu, and D. X. Feng, "Exact controllability for singular distributed parameter system in Hilbert space," *Sci. China Ser. F-Inf. Sci.*, vol. 52, pp. 2045–2052, 2009.

[16] V. E. Fedorov and B. Shklyar, "Exact null controllability of degenerate evolution equations with scalar control," *Sbornik: Math.*, vol. 203, pp. 1817–1836, 2012.

[17] E. S. Dzektser, "Generalization of the equation of motion of ground waters with a free surface," *English Transl. Soviet Phys. Dokl*, vol. 17, pp. 108–110, 1972.

[18] J. S. Respondek, "Numerical analysis of controllability of diffusive–convective system with limited manipulating variables," *Int. Commun. Heat Mass Transfer*, vol. 34, pp. 934–944, 2007.

[19] J. S. Respondek, "Numerical simulation in the partial differential equation controllability analysis with physically meaningful constraints," *Math. Comput. Simul.*, vol. 81, pp. 120–132, 2010.

[20] I. M. Gelfand and G. E. Shilov, *Generalized Function.* New York, NY, USA: Academic, 1964, vol. I.

[21] R. A. Adams, *Sobolev Spaces*. New York, NY, USA: Academic, 1975.